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An inequality relating mass and electric charge in general relativity

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Abstract. We show that the total mass and electric charge of an isolated gravitating system satisfy the inequality $m \geqslant |e|$, provided all matter eventually becomes enclosed within a single trapped surface.

A well known consequence of cosmic censorship and the so-called no hair theorems in general relativity is that the total mass m and electric charge e of an isolated, charged gravitating system satisfy the inequality

$$m \geqslant |e|, \tag{1}$$

provided all matter eventually becomes enclosed within a single trapped surface. In this paper we shall present a simple direct proof of this inequality which does not use cosmic censorship. We use a Witten-type argument similar to that recently used by Gibbons and Hull (1982) except that our spinor propagation law is based on a null, rather than a space-like, hypersurface. This greatly simplifies our analysis and allows us to use the highly developed mathematical machinery associated with a spin-coefficient formalism based on a null hypersurface. In this way we also avoid the necessity for complicated existence theorems for elliptic partial differential equations on which proofs based on Witten-type arguments usually rely. The use of null hypersurfaces in the present context does, however, have the drawback that it leads to the inequality (1) where m is the Bondi mass with respect to past null infinity $\mathscr{I}^-$ and, in order for this mass to be well defined, we must assume a certain degree of asymptotic flatness at $\mathscr{I}^-$. Furthermore, if we wish (1) to hold for the ADM mass at space-like infinity, i^0, we must assume a certain degree of regularity in the region of i^0, in which case (Ashtekar *et al* 1979)

$$m_{\mathrm{ADM}} \geqslant m \tag{2}$$

and hence

$$m_{\mathrm{ADM}} \geqslant |e|. \tag{3}$$

Inequality (2) is essentially a consequence of the Bondi mass-gain formula on $\mathscr{I}^-$.

Our main result is given as follows.

Theorem. Let M be a space-time which is asymptotically flat at $\mathscr{I}^-$ and contains a trapped surface T (homeomorphic to S^2) which may be connected to $\mathscr{I}^-$ by means of a regular, matter free, null hypersurface N. Then $m \geqslant |e|$ where m is the Bondi mass at the advanced time defined by N and e is the total electric charge contained within T.

The condition that N is matter free and extends to past null infinity guarantees that all (non-electromagnetic) matter is contained within the trapped surface. Using units such that $G = 1$, this implies that

$$\varphi_{ABA'B'} = \varphi_{AB}\bar{\varphi}_{A'B'} \qquad \text{and} \qquad \Lambda = 0 \tag{4}$$

on N, where $\varphi_{ABA'B'}$ and Λ are the spinor components of the Ricci curvature, and φ_{AB} is the Maxwell field.

Since our proof can most easily be expressed in terms of the Geroch–Held–Penrose (GHP) (1973) spin-coefficient notation, we start by introducing a GHP-type spinor dyad (o_A, ι_A) $(o_A\iota^A = 1)$ on N.

Let l^a be a past-pointing null vector field on N which points along the generators of N and satisfies $Þl_a = 0$ $(Þ = l^a\nabla_a)$, and let r be an affine parameter along the generators such that $Þr = 1$. Using l_a and r, we choose o_A such that $l_a = o_A\bar{o}_{A'}$ where $Þo_A = 0$, and ι_A such that $n_a := \iota_A\bar{\iota}_{A'}$ is orthogonal to the $r =$ constant cross sections. Under these conditions o_A and ι_A are defined up to

$$o_A \to a o_A \qquad \text{and} \qquad \iota_A \to a^{-1}\iota_A \tag{5}$$

on any $r =$ constant cross section.

The GHP spin-coefficients corresponding to such a dyad satisfy the relations

$$\begin{aligned} &\kappa = \varepsilon = 0 \qquad && \rho - \bar{\rho} = \rho' - \bar{\rho}' = 0 \\ &\tau + \bar{\tau}' = 0 \qquad && \tau + \bar{\beta}' - \beta = 0. \end{aligned} \tag{6}$$

Also, if $d\Omega$ represents the area element of the $r =$ constant cross sections of N we have

$$Þ\, d\Omega = -2\rho\, d\Omega \tag{7}$$

$$\oint ð\eta\, d\Omega = -\tfrac{1}{2}(p+q)\oint \eta\tau\, d\Omega \tag{8}$$

if η has weight (p, q) and $p - q = -2$. In equation (8) $ð$ is the GHP 'edth' operator which, when acting on a quantity with non-zero conformal wieght (i.e. $p + q \neq 0$), has a component proportional to τ which is not intrinsic to the $r =$ constant cross section; hence the non-vanishing of the right-hand side of equation (8). Similarly, for the $ð'$ operator, we have

$$\oint ð'\, d\Omega = -\tfrac{1}{2}(p+q)\oint \eta\bar{\tau}\, d\Omega \tag{9}$$

if η has weight (p, q) and $p - q = 2$.

Two quantities which play an important role in our proof are the divergences ρ and ρ'. These are real and transform according to

$$\rho \to a\bar{a}\rho \qquad \rho' \to (a\bar{a})^{-1}\rho' \tag{10}$$

under (5). If r is chosen to be constant on T, we have, by the trapped surface condition,

$$\rho \leqslant 0 \qquad \text{and} \qquad \rho' \leqslant 0 \qquad \text{on } T. \tag{11}$$

Furthermore, by asymptotic flatness at $\mathscr{I}^-$, we have

$$\rho = -1/r + O(r^{-2}) \tag{12}$$

and from equation (4) and equation (2.22) of GHP we have

$$Þ\rho = \rho^2 + \sigma\bar{\sigma} + \varphi_0\bar{\varphi}_0 \geqslant 0. \tag{13}$$

These two equations imply that $\rho < 0$ over the whole of N and, in particular, that $\rho \neq 0$.

Consider now two spinor fields λ_A and μ_A on N which are restrained to satisfy the following propagation equations:

$$o^B o^{A'}(\nabla_{AA'}\lambda_B + \varphi_{AB}\mu_{A'}) = 0 \tag{14}$$

$$o^B o^{A'}(\nabla_{AA'}\mu_B - \varphi_{AB}\lambda_{A'}) = 0 \tag{15}$$

These two equations are generalisation of the propagation equation

$$o^B o^{A'}\nabla_{AA'}\lambda_B = 0 \tag{16}$$

which we used in an earlier paper (Ludvigsen and Vickers 1982) and correspond closely to the Gibbons–Hull (1982) generalisation of Witten's equation (Witten 1981). In terms of the GHP notation, equations (14) and (15) are given by

$$Þ\lambda_0 + \varphi_0\bar{\mu}_0 = 0 \tag{17}$$

$$ð'\lambda_0 + \rho\lambda_1 + \varphi_1\bar{\mu}_0 = 0 \tag{18}$$

$$Þ\mu_0 - \varphi_0\bar{\lambda}_0 = 0 \tag{19}$$

$$Þ'\mu_0 + \rho\mu_1 - \varphi_1\bar{\lambda}_0 = 0 \tag{20}$$

where $\lambda_0 = \lambda_A o^A$, $\lambda_1 = \lambda_A \iota^A$, $\mu_0 = \mu_A o^A$ and $\mu_1 = \mu_A \iota^A$. From the form of these equations, plus the fact that $\rho \neq 0$, it is clear that λ_A and μ_A are uniquely determined over the whole of N if λ_0 and μ_0 are specified on some r = constant cross section.

Consider next the quantity

$$I(S) = -\oint [\rho(\lambda_1\bar{\lambda}_1 + \mu_1\bar{\mu}_1) + \rho'(\lambda_0\bar{\lambda}_0 + \mu_0\bar{\mu}_0)]\, d\Omega \tag{21}$$

where S is any r = constant cross section. From equation (10) we see that $I(S)$ invariant under transformation (5) and is therefore a functional only of S and the spinor fields λ_A and μ_A on S. Furthermore, by the trapped surface condition (11) we have

$$I(T) \geqslant 0. \tag{22}$$

We shall now proceed to show that our propagation equations imply that

$$I(S) \geqslant I(T) \tag{23}$$

for any cross section S lying in the past of T.

Since r is defined up to $r \to Ar + B$ ($A > 0$) we can choose it such that it takes a positive constant value on S and is zero on T. With this choice of r it is clear that (23) holds if

$$ÞI = dI/dr \geqslant 0. \tag{24}$$

After a long but straightforward spin-coefficient calculation involving equations (17)–(20), (4), (7), the GHP equations (2.22), (2.26), (2.31), (2.32) and (2.39), and integrating by parts using equations (8) and (9), we obtain

$$þI = \oint (X\bar{X} + Y\bar{Y})\, d\Omega \geqslant 0 \tag{25}$$

where

$$X = (\eth\lambda_0 + \sigma\lambda_1 + \varphi_0\bar{\mu}_1) \tag{26}$$

$$Y = (\eth\mu_0 + \sigma\mu_1 - \varphi_0\bar{\lambda}_1). \tag{27}$$

The inequality (23) is therefore automatically satisfied and, when combined with (22), gives

$$I(S) \geqslant 0. \tag{28}$$

In the next and final step of our proof we shall show that the fields λ_A and μ_A can be chosen such that

$$\lim I = m \pm e \tag{29}$$

where m is the Bondi mass at the advanced time defined by N. When combined with (28) this gives the required result, namely

$$m \geqslant |e|. \tag{30}$$

For the purpose of proving (29) it is convenient to take r to be a Bondi-type coordinate such that the $r = \text{constant}$ cross sections tend, asymptotically, to a metric two-sphere and such that

$$\rho = -1/r + O(r^{-3}). \tag{31}$$

With this choice of r, asymptotic flatness at $\mathscr{I}^-$ implies (Exton *et al* 1969)

$$\varphi_0 = \overset{0}{\varphi}_0 r^{-3} + O(r^{-4}) \qquad \varphi_1 = \overset{0}{\varphi}_1 r^{-2} + O(r^{-3}) \tag{32}$$

$$\overset{0}{\psi}_2 = \overset{0}{\psi}_2 r^{-3} + O(r^{-4}) \tag{33}$$

$$\rho = -r^{-1} + \sigma^0\bar{\sigma}^0 r^{-3} + O(r^{-5}) \tag{34}$$

$$\rho' = r^{-1} + (\sigma^0\dot{\bar{\sigma}}^0 + \eth_0^2\bar{\sigma}^0 + \overset{0}{\psi}_2)r^{-2} + O(r^{-3}) \tag{35}$$

$$\sigma = \sigma^0 r^{-2} + O(r^{-3}) \tag{36}$$

$$d\Omega = r^2\, d\Omega_0 - \sigma^0\bar{\sigma}^0\, d\Omega_0 + O(r^{-1}) \tag{37}$$

$$\eth = -r^{-1}\eth_0 + O(r^{-2}) \qquad \eth' = -r^{-1}\bar{\eth}_0 + O(r^{-2}) \tag{38}$$

where $d\Omega_0$ is the area element of a unit two-sphere and $\eth_0$ and $\bar{\eth}_0$ are the standard Newman–Penrose (1966) 'edth' operators. (The presence of the term $\eth_0^2\bar{\sigma}^0$ in equation (35), which does not appear in the corresponding expression in Exton *et al* (1969), is due to our different choice of ι^A.) In terms of these asymptotic quantities, m and e are given by

$$m = -\oint (\overset{0}{\psi}_2 + \sigma^0\dot{\bar{\sigma}}^0)\, d\Omega_0 \tag{39}$$

$$e = -\oint (\overset{0}{\varphi}_1 + \overset{0}{\bar{\varphi}}_1)\, d\Omega_0 \tag{40}$$

(Exton *et al* 1969).

From the above asymptotic relations it can easily be deduced that the fields λ_A and μ_A have the asymptotic form

$$\lambda_0 = \lambda_0^0 + O(r^{-2}) \qquad \mu_0 = \mu_0^0 + O(r^{-2}) \tag{41}$$

$$\lambda_1 = \lambda_1^0 + \varphi_1^0 \bar{\mu}_0^0 r^{-1} + O(r^{-2}) \tag{42}$$

$$\mu_1 = \mu_1^0 - \varphi_1^0 \bar{\lambda}_0^0 r^{-1} + O(r^{-2}), \tag{43}$$

where

$$\bar{\eth}_0 \lambda_0^0 = -\lambda_1^0 \qquad \bar{\eth}_0 \mu_0^0 = -\mu_1^0 \tag{44}$$

and where λ_0^0 and μ_0^0 can be chosen arbitrarily. We now restrict λ_0^0 and μ_0^0 such that

$$\lambda_0^0 = \bar{\mu}_1^0 \qquad \lambda_1^0 = -\bar{\mu}_0^0 \tag{45}$$

$$\lambda_0^0 \bar{\lambda}_0^0 + \mu_0^0 \bar{\mu}_0^0 = \lambda_0^0 \mu_1^0 - \lambda_1^0 \mu_0^0 = 1. \tag{46}$$

Equation (45) determines λ_A and μ_B up to a constant factor and equation (46), which is equivalent to $\lim_{r\to\infty} \lambda_A \mu^A = 1$, fixes this constant factor. Equations (45) and (46) therefore determine λ_A and μ_A uniquely over N.

If we now substitute these asymptotic relations into (21) and use equations (45), (46), (39) and (40), we obtain

$$\lim_{r\to\infty} I = -\oint (\dot{\bar{\sigma}}^0 \sigma^0 + \psi_2^0)\, \mathrm{d}\Omega_0 - \oint (\varphi_1^0 + \bar{\varphi}_1^0)\, \mathrm{d}\Omega_0$$

$$= m - e. \tag{47}$$

Thus, by inequality (28), we obtain

$$m - e \geqslant 0. \tag{48}$$

Finally, if we replace equation (45) by

$$\lambda_0^0 = -\bar{\mu}_1^0 \qquad \lambda_1^0 = \bar{\mu}_0^0$$

and repeat the above calculation, we obtain

$$m + e \geqslant 0 \tag{49}$$

and hence the required result, namely

$$m \geqslant |e|.$$

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References

Ashtekar A and Magnon-Ashtekar A 1979 *Phys. Rev. Lett.* **43** 181
Exton A R, Newman E T and Penrose R 1969 *J. Math. Phys.* **10** 1566

Geroch R, Held A and Penrose R 1973 *J. Math. Phys.* **14** 874
Gibbons G W and Hull C M 1982 *Phys. Lett.* **109B** 190
Ludvigsen M and Vickers J A G 1982 *University of York Preprint*
Newman E T and Penrose R 1966 *J. Math. Phys.* **7** 863
Witten E 1981 *Commun. Math. Phys.* **80** 381